Lecture 14

Thm $G = \bigcup_{w \in W} BwB$ and this union is disjoint.

Ex. $G = SL_n\mathbb{C}$. H = diag B = upper tri

Recall Gaussian elim: A invertible $\xrightarrow[\text{row ops}]{\text{"downward"}}$ Upper triang U

Means $L^{-1}A = U$, L^{-1} lower triang $\Rightarrow A = LU = \begin{pmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{pmatrix} U_1 \underbrace{\begin{pmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{pmatrix}}_{w_0} U_2$

$\Rightarrow Aw_0 \in Bw_0B$.

So you could apply this to Aw_0 to get $Aw_0^2 = A \in Bw_0B$.

But actually, this is only the generic situation.

The general statement has many slightly different forms – one is:

Using downward row ops, can give A the form of an upper tri matrix with its rows permuted

$L^{-1}A = P\mathcal{U} \Rightarrow A = LP\mathcal{U} \Rightarrow A = w_0 U_1 w_0 P U_2$

$\Rightarrow Aw_0 \in BwB$ where $w = w_0P$.

So in $SL_n\mathbb{C}$ case, the theorem is essentially G.E.

Note. Lower (B^-) and upper (B) tri matrices are both Borel subgroups of $SL_n\mathbb{C}$. They have the special property that

$B^- \cap B = H$ is a Cartan.

This is the smallest the int. can be (state soon!) & when this happens the product of the two Borels is open in G.

Note. Watch for the generalization of w_0 later!

Lie theory facts we'll use (Borel, Lin alg grps or others...)

① Borel subgroups are self-normalizing - check: where is this used?

② Maximal tori in a linear alg grp are conjugate
e.g. G semisimple or closed sub of.

└ closed subgrp iso to $(GL_1 k)^n$, $(\mathbb{C}^*)^n$ in this case

In a $\mathbb{C}$ semisimple group, a subgroup is a max torus iff it is a Cartan
(of form $N_G(\mathfrak{h})$, $\mathfrak{h} \subset \mathfrak{g}$ max abelian subalg of diag'ble)

So: H is a Cartan subgroup of G ✓
H is a max torus of G ✓

H is a Cartan subgroup of B ✗
H is a maximal torus of B ✓

③ $N_B(H) = H$.

④ The intersection of any two Borel subgroups of G contains a maximal torus. NOT EASY

Proof of thm. Recall we start with $H \subset B \subset G$, H max torus

Let $g \in G$. Then gBg^{-1} is a Borel.

$gBg^{-1} \cap B$ contains a maximal torus T

$\Rightarrow$ H, T are maximal tori in B $\Rightarrow$ $\exists\, b \in B$ s.t. $bHb^{-1} = T$

and gHg^{-1}, T are maximal tori in gBg^{-1} $\Rightarrow$ $\exists\, c \in gBg^{-1}$ s.t. $c(gHg^{-1})c^{-1} = T$

$c = gb'g^{-1}$ so $(gb')H(gb')^{-1} = T$

$N_B(H) = H \Rightarrow b$ and b' are unique up to right mult by elt H.

Now $H \xrightarrow[\text{conj } gb']{} T \xrightarrow[\text{conj } b^{-1}]{} H$

so $\quad b^{-1}gb' \in N(H) \quad$ or $\quad g = b\,\bar{w}\,(b')^{-1}$ where $\bar{w} \in N(H)$.

Moreover, the class $w \in W(G)$ of $\bar{w}$ in $N(H)/H$ is uniquely determined for

$$g = b\bar{w}(b')^{-1} \Rightarrow \underset{\bar{w}\in N(H).}{b\,h\,\bar{w}\,h'(b')^{-1}} = b\,\underbrace{\bar{w}\,h''h'}_{\text{another rep of } w.}(b')^{-1}$$

That is: $\forall\, g \in G \;\exists$ unique $w \in W$ s.t. $g \in BwB$. $\square$

<u>Cor</u>. $G/B = \bigcup_{w\in W} B\cdot w x_0$, i.e. $B \curvearrowright G/B$ with finitely many orbits.

For $w \in W$, let $C_w = Bwx_0$. This is called a Schubert cell.

Recall that in an alg group action, closure of an orbit is a union of orbits of smaller dimension. These closures are proj var.

<u>Def</u>. $X_w \subset G/B$ as $X_w = \overline{C_w}$ Schubert variety

So $X_w = \bigcup_{w \in \Gamma_w \subset W} C_w$.

Need: A) C_w homeo to $\mathbb{R}^{2n} \cong (\mathbb{D}^{2n})^{\circ}$ for some n.

B) $\forall\, w$, $(\mathbb{D}^{2n})^{\circ} \hookrightarrow G/B$ onto C_w extends to

$$\mathbb{D}^{2n} \longrightarrow G/B \text{ cts.}$$

Want: C) Description of Γ_w.

For $w \in W$ recall $w \cdot \Phi = \Phi$. Let $\Phi_w^- = \Phi^- \cap w\Phi^+$
$(w \cdot \Delta \neq \Delta ?)$
This is called the <u>inversion set</u> of w.

$\ell(w) = \# \Phi_w^-$ Note: If $\alpha, \beta \in \Phi_w^-$ and $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi_w^-$

Let $\mathcal{U}_w^- = \bigoplus_{\alpha \in \Phi_w^-} \mathfrak{g}_\alpha$. Lie subalg, nilpotent.

<u>General thm</u>. When a connected unipotent group acts on a proj variety, the orbits are affine spaces.

<u>Specific</u>. The orbit Bwx_0 is equal to $\mathcal{U}_w^- \cdot wx_0$ and the orbit map of $\mathcal{U}_w^-$ is injective. ($\Leftarrow$ loc inj)